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République Algérienne Démocratique Et Populaire
Université Mohamed BoudiafDe M'sila
Faculté Des Mathématiques Et De L'informatique
Département De Mathématiques

MEMOIRE DE FIN D'ETUDE

Présenté pour l'obtention du Diplôme de **MASTER**
Domaine : Mathématiques et Informatique
Filière: Mathématiques
Option : Analyse mathématique et numérique
Par
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Sujet

Les opérateur de Bernstein et leur propriétés

Date de soutenance :30-06-2019

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Promotion : 2018 / 2019

Dedication

Oh allah, the nighttime is pleasant only with your thank fullness, the daytime is pleasent only by your obediance, the moment are pleasent only by remembering you, and the heaven is not pleasent but by seeing you, bleased and exalted be you.

To him who proclaimed the mzssage, delivred up Alamanah and adviced the ommah, to the prophet of mercy Mohammed all prayers and blessings of Allah be upen him.

To him who I have his name with glory, my father.

To her, the symbol of love, tenderness and hard work that her prayers are the secret of my succes, to my mother whom with her presence I understand the meaning of life.

To my brothers and sisters.

To the angels Aya, Mohammed, Allaa, Soundoss, and Anes.

Acknowledgement

All perfect praise be to allah, the lord of the worlds and my the peace and blessing be upon our prophet Mohammed the example of educators, upon his household and his companious.

I present my thankfulness and expression of gratitude to mr.LAKHALI belkacem, who was very genrous with me by his advice and guidance.

To all who helped in accomplishment of this humble work.

To the board members who accepted to discuss and enrich this mimory.

Contents

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Introduction

In 1885, Weierstrass published his famous theorem asserting that every continuous function in a compact interval of the real line is the uniform limit of a sequence of algebraic polynomials. Several different proofs of Weierstrass's theorem are known, but a remarkable one was given by Bernstein in 1912.

it's very nice proof Weierstrass have proved his theorem in 8 pages but Bernstein in half a page and present that polynomials.

the function f is continous in$[0,1]$,so it is bounded: there existe $M>0$ such that $|f(x)| \leq M$ for every $x \in [0,1]$ .because f is continous in $[0,1]$ .so f is uniformly continous (Heine theoreme).say, for every $\varepsilon > 0$, there existe $\delta > 0$ such that, for every x and $y \in [0,1]$, $|x-y| \leq \delta$ implies $|f(x)-f(y)| \leq \varepsilon$.

Proof. let's give $\varepsilon > 0$ and $\delta > 0$ corresponding. for any $x \in [0,1]$ we have:

$$|f(x) - B_n(f)(x)| = \left| \sum_{k=0}^{n} (f(x) - f\left(\frac{k}{n}\right)) p_{n,k}(x) \right| \leq$$

$$\sum_{|x-k/n| \leq \delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| p_{n,k}(x) + \sum_{|x-k/n| > \delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| p_{n,k}(x)$$

the first sum is increased by $\sum\limits_{k=0}^{n} \varepsilon p_{n,k}(x)$ and the second sum by

$$\sum_{|x-k/n|>\delta} 2M p_{n,k}(x) \leq \frac{2M}{\delta^2} \sum_{|x-k/n|>\delta} \left(x - \frac{k}{n}\right)^2 p_{n,k}(x) \leq \frac{2Mx(1-x)}{n\delta^2} \leq \frac{M}{2n\delta^2}$$

so

$$|f(x) - B_n(f)(x)| \leq \varepsilon + \frac{M}{2n\delta^2} \leq 2\varepsilon$$

as soon $n \geq \left[\frac{M}{2\varepsilon\delta^2}\right] + 1$ ($[x]$ is the entire number of x).

like this last quantity is independente in $x \in [0,1]$,we proved the uniforme convergence in this intervalle. ∎

Bernstein operators became popular for several reasons: (1) they are given explicitly and depend only on the values of a function for rational values of the variable, (2) they have various shape-preserving properties and they provide the simplest means for the study of some problems, and (3) they (as well as their derivatives and integrals) are easy to handle in computer algebra systems and these are very useful when the evaluation of f is difficult and time-consuming.

the objective of this memory is to present this operators and some properties wich have studied until now.

Chapter 1

Functional spaces

1.1 Banach space

Definition 1 *(Norm)*

A norm on a linear space E is a function $\|.\|$: $E \to [0, \infty)$ such that

1. $\|x\| \geq 0$ *for all* $x \in E$, *and* $\|x\| = 0$ *implies that* $x = 0$.
2. $\|\alpha x\| = |\alpha| \, \|x\|$ *for all* $x \in E$ *and all* $\alpha \in \mathbb{C}$.
3. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in E$.

Example 2 *(the supermum norm)*

the application difined as:
$\forall f \in C([a,b], E), \|f\|_\infty = \sup\limits_{a \leq t \leq b} (|f(t)|)$ *is a norm in* $C([a,b], E)$.

1.1.1 Normed space

A normed space is simply a linear space that has a norm defined on it.

1.1.2 Convergence in normed spaces

Definition 3 *A sequence x_n in a normed space E is said to be a converge to a point $a \in E$ if for each $\varepsilon > 0$ there is an $n \in N$ such that $\|x_n - a\| < \varepsilon$ for all $n \geq N$.*

1.1.3 Cauchy sequence

A sequence x_n in a normed space is said to be a cauchy sequence if

$$\forall\varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall p, q \geq N_\varepsilon, \|x_p - x_q\| < \varepsilon$$

Notation 4 *clearly evry convergent sequence is also a cauchy sequence but the converse is not always true, this lead to the following result.*

1.1.4 Complete space

Definition 5 *A normed space $(E, \|.\|)$ is said to be a complete space, if any cauchy sequence is convergent sequence in E.*

Definition 6 *(Banach space)*

A Banach space is a normed space which is complete.

1.1.5 Examples

1. the popular banach space is the space of all continous functions with the the supermum norm $E = C([a, b], \|.\|_\infty)$.

2. evry finite dimensional normed space is banach space .

Remark 7 *in this memory we will focus on the banach space $E = C([a, b], \|.\|_\infty)$* .

Definition 8 *Two extended real numbers $p, q \in [1, \infty]$ are said to be conjugate exponents if*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $p = 1$, then we understand this to mean that $q = \infty$, and vice versa.

Proposition 9 *(Holders inequality)*

Let $p, q \in [1, \infty]$be conjugate exponents. Then for any two (finite or infinite) sequences $x_1, x_2, ...$and $y_1, y_2, ...$

$$\sum_k |x_k y_k| \leq \|x\|_p \|y\|_q .$$

Proposition 10 *(Minkowski's inequality)*

For every $p \in [1,\infty]$ *and every two (finite or infinite) sequences* $x_1, x_2, \ldots$ *and* $y_1, y_2, \ldots$

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p .$$

. In particular, if $x, y \in \ell^p$*, then* $x+y \in \ell^p$*.*

1.2 Hilbert space

1.2.1 Inner Product Spaces

Definition 11 *(Inner product space)*

Let E *be a complex vector space, A mapping* $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{C}$ *is called an inner product in* E *if for any* $x, y, z \in E$ *and* $\alpha, \beta \in \mathbb{C}$ *the following conditions are satisfied:*

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *(the bar denotes the complex conjugate).*
2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$.
3. $\langle x, x \rangle \geq 0.$
4. $\langle x, x \rangle = 0$ *implies* $x = 0.$

A vector space with an inner product is called an inner product space.

Definition 12 *(Norm in an inner product space)*

By the norm in an inner product space E *we mean the functional defined by* $\|x\| = \sqrt{\langle x, x \rangle}$.

1.2.2 Hilbert space

Definition 13 *(Hilbert space)*

A complete inner product space is called a Hilbert space.

Example 14 .

1. *Since* $\mathbb{R}$,*is complete, it is a Hilbert space, and so is* $\mathbb{R}^n$.
2. $L^2([a,b]) = \left\{ f : [a,b] \to \mathbb{R} / \int\limits_a^b f(x)^2 \, dx < \infty \right\}$*is a hilbert space with the norme* $\|f\|_2 = \left(\int\limits_a^b f(x)^2 \, dx \right)^{\frac{1}{2}}$.

Chapter 2

Linear operators

Definition 15 *A mapping A from X to Y, denoted as $A : X \to Y$, is called a linear operator (linear mapping, linear transformation) if for all x and y in the domain of A (defined below) and $\alpha,\beta \in \mathbb{C}$,*

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay,$$

where $Ax, Ay \in Y$

2.1 Continuous operators

Definition 16 *A linear operator $A : H_I \to H_2$ is said to be continuous at $x_o \in H_I$ if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that $\|Ax - Ax_o\| < \varepsilon$ whenever $\|x - x_o\| < \delta$.*

Theorem 17 *If $A : H_I \to H_2$ is a continuous linear operator at $x_0 \in H_I$, and if $x_n \to x_o$, then $Ax_n \to Ax_o$.*

Proof. Assume $x_n \to x$ (i.e., $\|x_n - x\| \to 0$) and show $\|Ax_n - Ax\| \to 0$ (i.e., $Ax_n \to Ax$). Indeed, $\|Ax_n - Ax\| = \|A(x_n - x)\| \leq \|A\| \|x_n - x\|$. Because $\|x_n - x\| \to 0$, then $\|Ax_n - Ax\| \to 0$, where we assume $\|A\| < \infty$. ■

2.2 Bounded operators

Definition 18 *A linear operator $A : H_I \rightarrow H_2$ is bounded on its domain if for all $x \in H_I$, there exist., a number $k > 0$such that*

$$\|Ax\|_2 \leq k\,\|x\|_1 \,.$$

Definition 19 *The norm of an operator $A : H_I \rightarrow H_2$, denoted as $\|A\|$ is the smallest number k that satisfies $\|Ax\|_2 \leq k\,\|x\|_1$ for all $x \in H_I$. This can be stated as*

$$\|A\| = \sup_{\|x\|_1 \neq 0} \frac{\|Ax\|_2}{\|x\|_1} = \sup_{\|x\|_1 \leq 1} \|Ax\|_2 = \sup_{\|x\|_1 = 1} \|Ax\|_2 \cdot$$

Theorem 20 *A linear operator $A : H_I \rightarrow H_2$ is continuous if and only if it is bounded.*

Proof. Assume A is bounded and x_0 an arbitrary point in H_I . Then $\|Ax - Ax_0\| = \|A(x - x_0)\| \leq k\,\|x - x_0\| \cdot$ Thus if $\|x - x_0\| < \delta$, setting $\delta = \varepsilon/k$ gives $\|A(x - x_0)\| < \varepsilon$. Then A is continuous at an arbitrary point and so is continuous on its domain. Now assume A is continuous, and in particular take $x_0 = 0$. Then for $\varepsilon = 1$ there exists a $\delta > 0$ such that $\|Ax\| < 1$ whenever $\|x\| < \delta$. For $x \neq 0$ let $z = \beta x$ with $\beta = \delta / \|x\|$ Then$\|z\|$ $= \|\beta x\| = \|\delta x / \|x\|\| = \delta$. Therefore, $1 > \|Az\| = \|A(\beta x)\| = |\beta|\,\|Ax\|$ and $\|Ax\| < 1/\beta = (1/\delta)\,\|x\|$and so A is bounded, Of course, for $x = 0$, $\|A0\| = 0 < \|x\|$. ■

Theorem 21 *If a linear operator $A : H_I \rightarrow H_2$ is continuous at one point, it is continuous on its domain.*

Proof. See [6] ■

2.3 Compact operator

Definition 22 *A bounded linear operator $A : H_I \rightarrow H_2$ is compact if, for each bounded sequence $\{x_n\} \in H_I$, there is a subsequence $\{x_{n_i}\}$ such that $\{Ax_{n_i}\}$converges in H_2*

Example 23 *1. Important examples of compact operators are integral operators T on $L^2[a,b]$ defined by*

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt.$$

where a and b are finite and k is continous.

2. *The identity operator I on an infinite dimensional Hilbert space H is not compact, although it is bounded. In fact, consider an orthonormal sequence (e_n) in H. Then the sequence $Ie_n = e_n$ does not contain aconvergent subsequence.*

Theorem 24 *Compact operators are bounded.*

Remark 25 *the converse of the theorem above is not necessarily true (the last example).*

2.4 Adjoint Operator

Definition 26 *.*
9

Suppose $T \in L(V,W)$ (V and W denote finite-dimensional inner product spaces)The adjoint of T is the function $T^* : W \to V$ such that

$$\langle Tv, w\rangle = \langle v, T^*w\rangle$$

for every $v \in V$ and every $w \in W$.

Proposition 27 *The adjoint is a linear map If*
$T \in L(V,W)$, then $T^* \in L(W,V)$.

Proof. Suppose $T \in L(V,W)$. Fix $w_1; w_2 \in W$. If $v \in V$, then

$$\begin{aligned}
\langle v, T^*(w_1+w_2)\rangle &= \langle Tv, w_1+w_2\rangle \\
&= \langle Tv, w_1\rangle + \langle Tv, w_2\rangle \\
&= \langle v, T^*w_1\rangle + \langle v, T^*w_2\rangle,
\end{aligned}$$

which shows that $T^*(w_1 + w_2) = T^*w_1 + T^*w_2$.

Fix $w \in W$ and $\lambda \in F$. If $v \in V$, then

$$\begin{aligned} \langle v, T^*(\lambda w)\rangle &= \langle Tv, \lambda w\rangle \\ &= \bar{\lambda}\,\langle Tv, w\rangle \\ &= \bar{\lambda}\,\langle v, T^*w\rangle \\ &= \langle v, \lambda T^*w\rangle \end{aligned}$$

which shows that $T^*(\lambda w) = \lambda T^*w$.

Thus T is a linear map, as desired. ■

Example 28 *Define* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *by*

$$T\left(x_1, x_2, x_3\right) = \left(x_2 + 3x_3, 2x_1\right)$$

Here T *will be a function from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$*. To compute* T *, fix a point* $(y_1; y_2) \in \mathbb{R}^2$ *Then for every* $(x_1, x_2, x_3) \in \mathbb{R}^3$ *we have*

$$\begin{aligned} \left\langle \left(x_1, x_2, x_3\right), T^*\left(y_1; y_2\right)\right\rangle &= \left\langle T\left(x_1, x_2, x_3\right), \left(y_1; y_2\right)\right\rangle \\ &= \left\langle \left(x_2 + 3x_3, 2x_1\right), \left(y_1; y_2\right)\right\rangle \\ &= x_2y_1 + 3x_3y_1 + 2x_1y_2 \\ &= \left\langle \left(x_1, x_2, x_3\right), \left(2y_2, y_1, 3y_1\right)\right\rangle \end{aligned}$$

Thus $T^*\left(y_1; y_2\right) = \left(2y_2, y_1, 3y_1\right)$.

2.5 Self-Adjoint Operator

Definition 29 *(self-adjoint)*

An operator $T \in L(V)$ *is called self-adjoint if* $T = T^*$*In other words,* $T \in L(V)$ *is self-adjoint if and only if*

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

for all $v; w \in V$.

2.6 Positive operator

Definition 30 *An operator $T \in L(V)$ is called positive if T is self-adjoint and*

$$\langle Tv, v\rangle \geq 0$$

for all $v \in V$.

If V is a complex vector space, then the requirement that T is self-adjoint can be dropped from the definition above because if V is a complex inner product space and $T \in L(V)$ Then T is self-adjoint if and only if $\langle Tv, v\rangle \in \mathbb{R}$.

Example 31 *Let k be a positive continuous function defined on $[a, b] \times [a, b]$. The integral operator T on $L^2([a, b])$ defined by*

$$(Tx)(s) = \int_a^b k(s,t)\, x(t)\, dt$$

is positive Indeed, we have

$$\langle Tx, x\rangle = \int_a^b \int_a^b K(s,t)\, x(t)\, \overline{x(t)} dtds = \int_a^b \int_a^b K(s,t)\, |x(t)|^2\, dtds \geq 0$$

for all $x \in L^2([a, b])$.

Theorem 32 *If A is an invertible positive operator, then its inverse $A^{-1}$is positive.*

Proof. see [5] . ■

Theorem 33 *Product of two commuting positive operators is a positive operator.*

Proof. see [5] . ■

Definition 34 *(square root)*

An operator R is called a square root of an operator T if $R^2 = T$.

Definition 35 *(eigenvalue)*

Suppose $T \in L(V)$, A number $\lambda \in F$ is called an eigenvalue of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Proposition 36 ***(Characterization of positive operators)***

1. *T is positive*

2. *T is self-adjoint and all the eigenvalues of T are nonnegative*

3. *T has a positive square root*

4. *T has a self-adjoint square root*

5. *there exists an operator $R \in L(V)$ such that $T = R^*R$.*

Proposition 37 *Every positive operator on V has a unique positive square root*

Proof. Suppose $T \in L(V)$ is positive. Suppose $v \in V$ is an eigenvector of T. Thus there exists λ such that $Tv = \lambda v$. Let R be a positive square root of T. We will prove that $Rv = \sqrt{\lambda} v$. This will imply that the behavior of R on the eigenvectors of T is uniquely determined. Because there is a basis of V consisting of eigenvectors of T , this will imply that R is uniquely determined. To prove that $Rv = \sqrt{\lambda} v$, note that the Spectral Theorem asserts that there is an orthonormal basis $e_1, ..., e_n$ of V consisting of eigenvectors of R. Because R is a positive operator, all its eigenvalues are nonnegative. Thus there exist nonnegative numbers $\lambda_1, ..., \lambda_n$ such that $Re_j = \sqrt{\lambda_j} e_j$ for $j = 1, ..., n$. Because $e_1, ..., e_n$ is a basis of V, we can write

$$v = a_1 e_1 + ... + a_n e_n$$

for some numbers $a_1, ..., a_n \in F$. Thus

$$Rv = a_1 \sqrt{\lambda_1} e_1 + ... + a_n \sqrt{\lambda_n} e_n$$

and hence

$$R^2 v = a_1 \lambda_1 e_1 + ... + a_n \lambda_n e_n.$$

But $R^2 = T$, and $Tv = \lambda v$. Thus the equation above implies

$$a_1 \lambda e_1 + ... + a_n \lambda e_n = a_1 \lambda e_1 + ... + a_n \lambda_n e_n$$

The equation above implies that $a_{j'}(\lambda - \lambda_j) = 0$ for $j = 1, ..., n$ Hence

$$v = \sum_{\{j:\lambda_j = \lambda\}} a_j e_j$$

and thus

$$Rv = \sum_{\{j:\lambda_j = \lambda\}} a_j \sqrt{\lambda j} e_j = \sqrt{\lambda} v$$

as desired.

Remark 38 *A positive operator can have infinitely many square roots (although only one of them can be positive). For example, the identity operator on V has infinitely many square roots if $dimV > 1$.*

■

2.6.1 General Estimates for Positive Linear Operators

Definition 39 *Fix $r \in \mathbb{N}$, a function $f : I \to \mathbb{R}$, and $h \geq 0$. For $x \in I$ the difference of order r of f in x, with step h, is defined by*

$$\overset{\rightarrow r}{\Delta}_h f(x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x + kh).$$

Definition 40 *If r is a positive integer and $f : I \to \mathbb{R}$ for $t > 0$ the usual modulus of continuity of order r of f is defined by*

$$\omega_r(f,t) = \sup_{h \in (0,t]} \sup_{x, x+rh \in I} \left| \overset{\rightarrow r}{\Delta}_h f(x) \right|$$

We also use the notation $\omega(f,t) = \omega_1(f,t)$.

In this section we present some known results related to positive linear operators. There is a Holder-type inequality for positive linear functionals. Let I be an interval and V a subspace of $C(I)$ such that $|f| \in V$, whenever $f \in V$.

Lemma 41 *If $L : V \to \mathbb{R}$ is a positive linear functional, then for all $p \in (1, \infty)$ and $f, g \in V$ satisfying $f, g \geq 0$ and $fg, f^p, g^p \in V$, one has*

$$L(fg) \leq (L(f^p))^{1/p}(L(g^q))^{1/q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Fix positive real numbers a and b and consider the function$F(t) = \frac{t^p}{p}\alpha^p + \frac{t^{-q}}{q}b^q, \qquad t > 0$
since $\lim\limits_{t \to 0} F(t) = \lim\limits_{t \to \infty} F(t) = \infty$ and $F'(t_0) = 0$ if and only if

$$t_0 = b^{q/(p+q)}a^{p/(p+q)} = b^{1/p}a^{-1/q}$$

for all $t > 0$ we have

$$F(t) \geq F(t_0) = \frac{1}{p}a^{p-p/q}b + \frac{1}{q}b^{q-q/p}a = \frac{1}{p}ab + \frac{1}{q}ab = ab. \tag{2.1}$$

Let us see that, for all $x \in I$ and all $t > 0$

$$f(x)g(x) \leq \frac{t^p}{p}f(x)^p + \frac{t^{-q}}{q}g(x)^q.$$

Of course, if $f(x)g(x) = 0$, it is trivial and, if $f(x)g(x) \neq 0$, the inequality follows from Eq (2.1) with $a = f(x)$ and $b = g(x)$.

Since L is a positive linear functional on V, we obtain

$$L(fg) \leq \inf_{t>0} \frac{t^p}{p}L(f^p) + \frac{t^{-q}}{q}L(g^q) = (L(f^p))^{1/p}(L(g^q))^{1/q}.$$

where for the last equality we have applied Eq. (4.1) with $a = (L(f^p))^{1/p}$ and $b = (L(g^q))^{1/q}$. ■

Proposition 42 *if $L : C[0,1] \to C[0,1]$ is a positive linear operator that reproduces linear functions, then*

$$|L(f,x) - f(x)| \leq \left(1 + \frac{\sqrt{L((e_1 - x)^2, x)}}{\delta}\right)\omega(f,\delta)$$

and (Shisha and Mond [9])

$$|L(f,x) - f(x)| \leq \left(1 + \frac{\sqrt{L((e_1 - x)^2, x)}}{\delta^2}\right)\omega(f,\delta)$$

for each $f \in C[0,1]$, $x \in [0,1]$, and $0 < \delta \leq 1$.

Proposition 43 *If $L : C[0,1] \to C[0,1]$ is a positive linear operator that reproduces linear functions,$f \in C^1[[0,1]]$, $x \in [0,1]$, and*
$0 < \delta \leq 1$,*then*

$$|L_n(f,x) - f(x)| \leq \left(\sqrt{L((e_1 - x)^2, x)} + \frac{1}{\delta} L((e_1 - x)^2, x) \right) \varpi(f', \delta).$$

2.6.2 Estimates Using the Second Derivative

Proposition 44 $[ref]$ [9 *p.202-203*] *If $L : C[0,1] \to C[0,1]$ is a positive linear operator that reproduces linear functions and*

$$B(M) = \{ f \in C[0,1] : f' \in AC[0,1] \ \ and \ \ \|f''\| \leq M \}.$$

then

$$\sup_{f \in B(M)} \|f - L(f)\|_\infty = \frac{M}{2} \left\| L((e_1 - xe_0)^2, x) \right\|_\infty$$

Proof. see[9] . ■

Chapter 3

Bernstein operators

3.1 introduction

The polynomials of bernstein introduced in 1912 in Bernstein's constructive proof of the Weierstrass approximation theorem. Since then they have been the object of multiple investigations, serving many times as a guide for several theorems in Approximation Theory. The Korovkin theorem [8] is a typical example. In this note, starting from several convergence results for the iterates of B_n.These operators are, very probably, the most studied linear **positive operators**. They were generalized and modified in a great number of variants, like the well-known Bernstein-type operators of Kantorovich, The advantages of the Bernstein operators consist in their simplicity, and on their sharp properties of approximation. From certain points of view the Bernstein operators play an extremal position in some classes of operators. We mention here only some basic properties of these operators.

3.2 Weierstrass Theorem

Theorem 45 *(Weierstrass)*

Let f be a continuous real-valued function on $[a, b]$. Then there is a sequence of polynomials (p_n) that converges uniformly to f on $[a, b]$. In the language of normed vector spaces, this theorem says that the polynomials are dense in $C[a, b]$ in the max norm. In fact, this theorem is sufficiently important that many different proofs have been found. The proof we give was found in 1912 by Bernstein, a Russian mathematician. It explicitly constructs the *approximating polynomial. by a change of variable, we can bring back our study to the interval*$[0, 1]$ *,what we will do now.*

Theorem 46 *Note*

$$p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-k)^{n-k}, 0 \leq k \leq n. \qquad \begin{pmatrix} n \\ k \end{pmatrix} = C_n^k.$$

these polynomial are a base of $\mathbb{R}_n[x]$ *and verified, for every* $x \in [0, 1]$ *, the following identities:*

1. $p_{n,k}(x) \geq 0.$
2. $\sum\limits_{k=0}^{n} p_{n,k}(x) = 1.$
3. $\sum\limits_{k=0}^{n} k p_{n,k}(x) = nx.$
4. $\sum\limits_{k=0}^{n} k(k-1) p_{n,k}(x) = n(n-1)x^2.$
5. $\sum\limits_{k=0}^{n} (nx-k)^2 p_{n,k}(x) = nx(1-x).$

Proof. the independence of polynomials $x^k(1-x)^{n-k}, 0 \leq k \leq n$,proof with recurrence.

let $\alpha_0, \alpha_1 ... \alpha_{n+1}$ such as

$$\sum_{k=0}^{n+1} \alpha_k x^k (1-x)^{n+1-k} = 0$$

for every $x \in \mathbb{R}$ with $x = 1$ we get $\alpha_{n+1} = 0$ dividing by $1 - x$

$$\sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k} = 0$$

and we apply the hypothesis of recurrence.

1. the positivity of polynomials is obvious .

2. we use the binome formula:$(x+(1-x))^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}$.

3. $\sum_{k=0}^{n} k p_{n,k}(x) = nx \sum_{k=1}^{n} (\begin{matrix} n-1 \\ k-1 \end{matrix}) x^{k-1}(1-x)^{n-k} = nx \sum_{k=0}^{n-1} p_{n-1,k}(x) = nx.$

4. $\sum_{k=0}^{n} k(k-1) p_{n,k}(x) = n(n-1)x^2 \sum_{k=2}^{n} (\begin{matrix} n-2 \\ k-2 \end{matrix}) x^{k-2}(1-x)^{n-k} = n(n-1)x^2.$

5. we use the previous identities, $(nx-k)^2 = k(k-1) - (2nx-1)k + n^2x^2$.

■

Definition 47 *the bernstein polynomial of degree n of $f : [0,1] \to \mathbb{R}$ equal*

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-k)^{n-k}$$

Theorem 48 *(Bernstein)*
For every continous function $f : [0,1] \to \mathbb{R}$, the sequence $B_n(f)$ converge uniformly to f in $[0,1]$.

Proof. the function f is continous in$[0,1]$,so it is bounded: there existe $M > 0$ such that $|f(x)| \leq M$ for every $x \in [0,1]$.because f is continous in $[0,1]$.so f is uniformly continous (Heine theoreme).say, for every $\varepsilon > 0$, there existe $\delta > 0$ such that, for every x and $y \in [0,1]$, $|x-y| \leq \delta$ implies $|f(x) - f(y)| \leq \varepsilon$.

let's give $\varepsilon > 0$ and $\delta > 0$ corresponding. for any $x \in [0,1]$ we have:

$$|f(x) - B_n(f)(x)| = \left| \sum_{k=0}^{n} (f(x) - f\left(\frac{k}{n}\right)) p_{n,k}(x) \right| \leq$$

$$\sum_{|x-k/n| \leq \delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| p_{n,k}(x) + \sum_{|x-k/n| > \delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| p_{n,k}(x)$$

the first sum is increased by $\sum_{k=0}^{n} \varepsilon \, p_{n,k}(x)$ and the second sum by

$$\sum_{|x-k/n| > \delta} 2M p_{n,k}(x) \leq \frac{2M}{\delta^2} \sum_{|x-k/n| > \delta} \left(x - \frac{k}{n}\right)^2 p_{n,k}(x) \leq \frac{2Mx(1-x)}{n\delta^2} \leq \frac{M}{2n\delta^2}$$

we use theorem (45) and inequality $x(1-x) \leq 1/4$.obtained so

$$|f(x) - B_n(f)(x)| \leq \varepsilon + \frac{M}{2n\delta^2} \leq 2\varepsilon$$

as soon $n \geq \left[\frac{M}{2\varepsilon\delta^2}\right] + 1$. like this last quantity is independente in $x \in [0,1]$,we proved the uniforme convergence in this intervalle. ■

3.3 Representations for the Derivatives

Proposition 49 *If $n \in \mathbb{N}, x \in [0,1]$ and $f : [0;1] \to \mathbb{R}$, then*

$$\frac{x(1-x)}{n} B_n'(f,x) = B_n((e_1 - x)f(e_1), x)$$

where $e_q(t) = t^q$.

Proposition 50 *If $n \in \mathbb{N}, x \in [0,1]$ and $f : [0;1] \to \mathbb{R}$, then*

$$B_n'(f,x) = n \sum_{k=0}^{n-1} \left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right) p_{n-1,k}(x)$$

Proof. we have

$$\begin{aligned} B_n'(f,x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \left(\begin{array}{c} n \\ k \end{array} \right) \left[k x^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1} \right] \\ &= n \sum_{k=0}^{n-1} \left(f\left(\frac{k+1}{n}\right) - f(\frac{k}{n}) \left(\begin{array}{c} n-1 \\ k \end{array} \right) x^k (1-x)^{n-1-k} \right) \end{aligned}$$

■

3.4 Bernstein Polynomials as Linear Operators

3.4.1 Estimates for the Norm of a Bernstein Operator

Recall that, if $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are normed spaces and $L : X \to Y$ is a continuous linear operator, then the norm of L is defined as

$$\|L\|_{X\to Y} = \sup\{\|L(x)\|_Y : \|x\|_X \leq 1\}$$

Moreover, a linear operator $L : X \to Y$ is continuous if and only if $\|L\|_{X\to Y} \leq 1$.

Proposition 51 *For each $n \in \mathbb{N}$, one has*

$$\|B_n\|_{C[0,1]\to C[0,1]} = 1$$

Proof. *It follows from the identity $B_n(e_0, x) = 1$ and the inequality $\|B_n(f)\| \leq \|f\|$, $f \in C[0,1]$.*

If in the space $C^1[0,1]$. we consider the norm

$$\|g\|_1 = \max\{\|g\|, \|g'\|\}$$

as the following proposition shows,$\{B_n\}$ $(B_n : C^1[0,1] \to C^1[0,1])$ is a uniformly bounded sequence of linear operators. ■

Proposition 52 *If $n > 1$, then $B_n : C^1[0,1] \to C^1[0,1]$ is a bounded operator and*

$$\|B_n\|_{C^1[0,1]\to C^1[0,1]} = 1$$

Proposition 53 *If $n \in \mathbb{N}$ and $f \in C_\varrho[0,1]$, then $B_n(f) \in C_\varrho[0,1]$ and*

$$\|B_n\|_{C_\varrho[0,1]\to C_\varrho[0,1]} \leq 1$$

Theorem 54 *Fix $0 \leq a < c < d < b \leq 1$.*

Assume $f \in C[0,1]$ and there exist $\alpha, \beta \in \mathbb{R}$ and a bounded and measurable function $h : [c,d] \to \mathbb{R}$ so that

$$f(x) = \alpha x + \beta + \int_c^x \int_c^t h(s)dsdt, \qquad x \in [c,d].$$

There exists a constant C such that, for every $n \in \mathbb{N}$,

$$|B_n(f,x) - f(x)| \leq \frac{C}{n}, \qquad x \in [c,d]$$

Proof. Fix δ, $0 < \delta < (b-a)/4$. Let g be an infinitely differentiable function defined on $[0,1]$ that satisfies the following:

$$\begin{aligned} g &\equiv 1, \qquad x \in [a+2\delta, b-2\delta], \\ g &\equiv 0, \qquad x \in [a+\delta, b-\delta], \end{aligned}$$

and

$$0 \le g \le 1, \qquad x \in [0,1],$$

Then

$$|B_n(f,x) - f(x)| \le |B_n(f,x) - B_n(fg,x)| + |B_n(fg(x) - B_n(fg)(x)|.$$

We shall estimate the two terms on the right separately.

Since g is infinitely differentiable and zero outside of $[a+\delta$,$b-\delta]$ has a continuous derivative .$(gf)'$ that satisfies a Lipschitz condition

$$|(gf)'(b) - (gf)'(a)| \le K\,|b-a|,$$

for some$K$.

$$|B_n(gf,x) - gf(x)| \le \frac{Kx(1-x)}{n} \le \frac{K}{4n}, \qquad 0 \le x \le 1.$$

For $x \in [a+3\delta, b-3\delta]$,if we set $Q = \{k : k/n \notin [a+2\delta, b-2\delta]\}$ .then

$$|B_n(f,x) - B_n(fg,x)| \le \sum_0^n \left| f\left(\frac{k}{n}\right) - (fg)\left(\frac{k}{n}\right) \right| p_{n,k}(x)$$

$$\le 2\,\|f\| \sum_{x \in Q} p_{n,k}(x) \le 2\,\|f\| \sum_{|k/n-x|>\delta} p_{n,k}(x)$$

$$\le \frac{2}{\delta^2}\,\|f\| \sum_{|k/n-x|>\delta} \left(\frac{k}{n} - x\right)^2 p_{n,k}(x) \le \frac{1}{n}\frac{1}{\delta^2}\,\|f\|.$$

■

3.5 Estimates for Lipschitz Functions

there are special classes of functions for which better estimates can be given. Several proofs are known for the remainder in approximations of Lipschitz functions by Bernstein operators. Kac considered these classes and, later he recognized that a more general result was proved previously by Popoviciu.

Proposition 55 *Fix* $\alpha \in [0,1]$, $M > 0$*and* $f \in C\,[0,1]$*such that*

$$|f(x) - f(y)| \leq M\,|x-y|^{\alpha}\,, \qquad x, y \in [0,1]\,.$$

For each $n \in \mathbb{N}$ *and* $x \in [0,1]$, *one has*

$$|f(x) - B_n(f,x)| \leq M\,(x(1-x)/n)^{\alpha/2}\,.$$

Proof. From Lemma (37) we obtain

$$|f(x) - B_n(f,x)| \leq B_n\left(|f(t) - f(x)|\,, x\right) \leq MB_n\left(|t-x|^{\alpha}\,, x\right)$$

$$\leq M\left(B_n\left((t-x)^2, x\right)\right)^{\alpha/2} = M\,(x(1-x)/n)^{\alpha/2}\,.$$

■

3.6 Bernstein Polynomials and Convex Functions

Definition 56 *Fix* $r \in N$ *and* $f : [0,1] \to \mathbb{R}$. *The function* f *is said to be convex of order* r *on* $[0,1]$, *if all its divided differences of order* $r+1$, *on* $r+2$ *distinct points of* $C\,[0,1]$, *are positive.*

Theorem 57 *For any* $f \in C\,[0,1]$ *the following assertions are equivalent:*
(i) f *is convex.*
(ii) *For every* $n \in \mathbb{N}$, $B_n(f)$*is convex.*
$(iii)$$B_{n+1}(f,x) \leq B_n(f,x)$*for every* $n \in \mathbb{N}$ *and* $x \in [0,1]$.
$(iv)$$f(x) \leq B_n(f,x)$*for every* $n \in \mathbb{N}$ *and* $x \in [0,1]$.
(v) *For each* $x \in [0,1]$,

$$\lim_{n\to\infty} \sup n\left(B_n(f,x) - f(x)\right) \geq 0.$$

3.7 The operator A_n and the Adjoint Operator

In this section we present some ideas of de Leeuw .

For $f \in C[0,1]$ and $n \geq 1$ set

$$A_n(f,x) = \sum_{k=1}^{n-1} \left(n \int_{-1/2n}^{1/2n} f\left(\frac{k}{n}+t\right) \right) p_{n,k}(x). \tag{3.1}$$

Recall that $C_0^2[0,1]$ is the space of all $f \in C^2[0,1]$ vanishing in neighborhoods,of the endpoints of $[0,1]$. For functions in $C_0^2[0,1]$the operators A_n are closely connected to the Bernstein ones.

Lemma 58 *Assume $f \in C[0,1]$, $0 < a < c < d < b < 1$, and A_n is defined by Eq (3.1). If for $x \in [a,b]$ and $h > 0$*

$$|f(x+h) - 2f(x) + f(x+h)| \leq Mh,$$

then there exists a constant $C(c,d)$ such that

$$|A_n(f,x) - B_n(f,x)| \leq \frac{C(c,d)}{n}, \qquad x \in [c,d].$$

Proof. *Choose $\delta > 0$ such that $[c,d] \subset [a+\delta, b-\delta]$.*

For $x \in [a+\delta, b-\delta] = I$,

$$|A_n(f,x) - B_n(f,x)| \leq$$

$$\sum_{k/n\in I} \left| n \int_{-1/2n}^{1/2n} \left(f\left(\frac{k}{n}+t\right) - f(\frac{k}{n}) \right) dt \right| p_{n,k}(x) + 2\|f\| \sum_{k/n\notin I} p_{n,k}(x)$$

$$\leq \sum_{k/n\in I} \left| n \int_{0}^{1/2n} \left(f\left(\frac{k}{n}+t\right) - 2f(\frac{k}{n}) + f\left(\frac{k}{n}-t\right) \right) \right| p_{n,k}(x)$$

$$+\frac{2}{\delta^2} 2\|f\| \sum_{k/n\notin I} \left(\frac{k}{n}-x\right)^2 p_{n,k}(x)$$

$$\leq n \sum_{k/n \in I} \int_0^{1/2n} Mtdtp_{n,k}(x) + \frac{\|f\|}{\delta^2 n} \leq \frac{M}{8n} + \frac{\|f\|}{\delta^2 n}$$

■

Lemma 59 *For $f \in C[0,1]$, for each $n \in \mathbb{N}$,*

$$\|A_n(f) - B_n(f)\|_{C[0,1]} \leq \frac{\|f''\|}{12n^2}.$$

Proof. Since $f(0) = f(1) = 0$,

$$|A_n(f,x) - B_n(f,x)| \leq \sum_{k=1}^{n-1} n \left| \int_{-1/2n}^{1/2n} \left(f(\frac{k}{n} + t) - f(\frac{k}{n}) \right) dt \right| p_{n,k}(x)$$

$$= \sum_{k=1}^{n-1} n \left| \int_{-1/2n}^{1/2n} \left(tf'(\frac{k}{n}) + \int_{k/n}^{t+k/n} (\frac{k}{n} + t - s) f''(s) ds \right) dt \right| p_{n,k}(x)$$

$$= \sum_{k=1}^{n-1} n \left| \int_{-1/2n}^{1/2n} \int_{k/n}^{t+k/n} (\frac{k}{n} + t - s) f''(s) ds dt \right| p_{n,k}(x)$$

$$\leq \frac{n}{2} \|f''\| \sum_{k=1}^{n-1} \int_{-1/2n}^{1/2n} t^2 dt p_{n,k}(x) = \frac{\|f''\|}{12n^2}$$

For $g \in L_1[0,1]$ and a bounded measurable function f we denote

$$\langle g, f \rangle = \int_0^1 g(t) f(t) dt. \tag{3.2}$$

If $\psi \in L_1[0,1]$ and $n \geq 1$, define

$$A_n^*(\psi, x) = \begin{cases} n(\psi, p_{n,k}), \left| x - \frac{n}{k} \right| \leq \frac{1}{2n}, & k = 1, 2, ..., n-1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

■

Proposition 60 *Let A_n and A_n^* be given by Eqs (3.1) and (3.3) respectively.*

(i) For all $g \in C_0^2[0,1]$ and $n \in \mathbb{N}$, one has

$$\left\| n(A_n(g)-g) - \frac{\varphi^2 g''}{2} \right\| \leq \frac{\|g''\|}{12n^2} + \frac{1}{6}\omega\left(g'', \sqrt{\frac{2}{n}}\right).$$

(ii) For all $g \in C_0^2[0,1]$ and each $\psi \in C_0^2[0,1]$,

$$\lim_{n\to\infty} \langle n(A_n(\psi)-\psi), g\rangle = \frac{1}{2}\left\langle \psi, \varphi^2 g''\right\rangle.$$

(iii) For all $f \in C[0,1]$ and each $\psi \in C_0^2[0,1]$,

$$\lim_{n\to\infty} \langle n(A_n(\psi)-\psi), f\rangle = \frac{1}{2}\left\langle (\varphi^2\psi'')'', f\right\rangle.$$

Lemma 61 *For all $\psi \in L_1[0,1]$ and $f \in C[0,1]$,*

$$\langle A_n^*(\psi), f\rangle = \langle \psi, A_n(f)\rangle$$

Proof. Set $Q_k = \{t \in [0,1] : |t-k/n| \leq 1/2n\}$. One has

$$\begin{aligned}
\langle A_n^*(\psi), f\rangle &= \int_0^1 A_n^*(\psi,t)f(t)dt = \sum_{k=1}^{n-1} n\langle \psi, p_{n,k}\rangle \int_{Q_k} f(t)dt \\
&= \int_0^1 \psi(s)\left(\sum_{k=1}^{n-1} n \int_{-1/2n}^{1/2n} f(k/n+t)dt\, p_{n,k}(s)\right) ds = \langle \psi, A_n(f)\rangle.
\end{aligned}$$

■

Lemma 62 *For $m = 0,1,2, n \geq 1$ and an irrational $x \in [1/2n, 1-1/2n]$,*

$$|A_n^*(e_m, x) - e_m(x)| \leq \frac{11}{n}$$

Lemma 63 *Fix $\psi \in C_0^2[0,1]$.*

(i) There exists a constant $K_1(f)$ such that, if $n \geq 1$ and $x \in [1/2n, 1-1/2n]$ is irrational,

$$|A_n^*(\psi, x) - \psi(x)| \leq \frac{K_1(f)}{n}.$$

(ii) There exists a constant $K_2(f)$ such that, if $n \geq 2$,

$$\|n(A_n^*(\psi)-\psi)\|_{L_1[0,1]} \leq K_2(f).$$

3.8 Some Convergence Problems

Absolute Convergence

If $f : [0,1] \to \mathbb{R}$ is a convex function, then the sequence$\{B_n(f,x)\}$ decreases (see Theorem 55), and therefore

$$\sum_{n=2}^{\infty} |B_n(f,x) - B_{n-1}(f,x)| < \infty$$

at every point x where $B_n(f,x)$ converges to $f(x)$. Absolute convergence was studied by Li.

Theorem 64 *if $f \in C^1[0,1]$ and $\sum_{n=1}^{\infty} \omega(f', 1/n) < \infty$,then*

$$\sum_{n=2}^{\infty} |B_n(f,x) - B_{n-1}(f,x)| \leq 2 \sum_{n=2}^{\infty} \frac{1}{n} \varpi(f', \frac{1}{n}).$$

for every $x \in [0,1]$.

Strong Convergence

Theorem 65 *For each real $p \geq 1$,$f \in C[0,1]$,$n \in N$, and $x \in [0,1]$, one has*

$$\left(\sum_{k=0}^{n} p_{n,k}(x) \left| f(\frac{k}{n}) - f(x) \right|^p \right)^{1/p} \leq \left(1 + C_p \sqrt{x(1-x)} \right) \omega \left(f, \frac{1}{\sqrt{n}} \right)$$

Proof. It follows from the properties of the first modulus of continuity and the Minkowski inequality that

$$\left(\sum_{k=0}^{n} p_{n,k}(x) \left| f(\frac{k}{n}) - f(x) \right|^p \right)^{1/p} \leq \left(\sum_{k=0}^{n} p_{n,k}(x) \left(\omega \left(f, \left| \frac{k}{n} - x \right| \right) \right)^p \right)^{1/p}$$

$$\leq \omega \left(f, \tfrac{1}{\sqrt{n}} \right) \left(\sum_{k=0}^{n} p_{n,k}(x) \left(1 + \sqrt{n} \left| \tfrac{k}{n} - x \right| \right)^p \right)^{1/p}$$
$$\leq \omega \left(f, \tfrac{1}{\sqrt{n}} \right) \left(\left(\sum_{k=0}^{n} p_{n,k}(x) \right)^{1/p} + \sqrt{n} \left(\sum_{k=0}^{n} p_{n,k}(x) \left| \tfrac{k}{n} - x \right|^p \right)^{1/p} \right)$$
$$\leq \omega \left(f, \tfrac{1}{\sqrt{n}} \right) \left(1 + \sqrt{n} C(q) \sqrt{\tfrac{x(1-x)}{n}} \right) \blacksquare$$

3.9 Approximations in Holder Norms

For $m \in \mathbb{N}_0$, denote by $C^m[0,1]$ the family of $m-$times continuously differentiable functions $f:[0,1] \to \mathbb{R}$. It is known that $C^m[0,1]$ is a Banach space with the norm

$$\|f\|_m = \sum_{k=0}^{m} \left\| D^k f \right\|,$$

where $D^0 f = f, D^1 f = f'$ and $D^{k+1} f = D^1(D^k(f))$.Let $\Phi[0,1]$ be the family of increasing concave functions $\phi : [0,1] \to \mathbb{R}$ such that $\phi(0) = 0, \phi(t) \geq 1$ for $t > 0$ and $\phi(1) = 1$. Notice that $\phi(t) = t^\alpha \in \Phi[0,1]$for $0 < \alpha \leq 1$.For each $\phi \in \Phi[0,1]$we consider the associated Holder-type space

$$Hol_\phi[0,1] = \{f \in C[0,1] : \omega(f,t) \leq K(f)\phi(t)\}.$$

For $f \in Hol_\phi[0,1]$ set

$$\theta_\phi(f,t) = \sup_{0<h\leq t} \frac{\omega(f,t)}{\phi(h)} \text{ and } \theta_\phi(f) = \theta_\phi(f,1)$$

We also need the following space:

$$hol_\phi[0,1] = \left\{ f \in Hol_\phi[0,1] \ : \lim_{t \to 0} \theta_\phi(f,t) = 0 \right\}$$

We will study simultaneous approximations in some Sobolev-type spaces. For $m \in \mathbb{N}_0$ and $\phi \in \Phi[0,1]$, consider the spaces

$$C_\phi^m[0,1] = \{f \in C^m[0,1] \ : D^m(f) \in Hol_\phi[0,1]\}$$

and

$$\breve{C}_\phi^m[0,1] = \{f \in C^m[0,1] \ : D^m(f) \in Hol_\phi[0,1]\}$$

with the norm

$$\|f\|_{m,\phi}^* = \|f\|_m + \theta_\phi(D^m(f)).$$

For completeness we also set

$$C_0^m[0,1] = C^m[0,1] \text{ and } \|f\|_{m,0}^* = \|f\|_m^*.$$

When $\phi(t) = t^\alpha$we simply write $\breve{C}_\phi^m[0,1]$ and $\theta_\alpha(f,t)$.

In the spaces $C_{\phi}^{m}[0,1]$ different equivalent norms can be considered and the constants in the estimates depend on the norm we use. Instead of the norm $\|f\|_{m,k}^{*}$ we will use

$$\|f\|_{m,0} = \|f\| + \left\|f^{(m)}\right\|$$

for $C_0^m[0,1]$, It is known that, for $m > 1$ fixed, there exists a constant A_m such that, for each $f \in C^m[0,1]$ and $1 \leq k < m$, one has

$$\left\|f^{(k)}\right\| \leq A_m \|f\| + \left\|f^{(m)}\right\|.$$

Thus the norms $\left\|f^{(m)}\right\|$ and $\|f\|_{m,0}$ are equivalent. It is known that, for $0 < \alpha < 1$, the closure of the polynomials in $C_{\alpha}^{0}[0,1]$ is $\overset{\smile}{C}{}_{\alpha}^{0}[0,1]$. Hence, in order to approximate by Bernstein polynomials we should restrict the analysis to the spaces $C_{\phi}^{m}[0,1]$.

Proposition 66 *If* $\phi \in \Phi[0,1]$, *then*

$$\frac{t}{\phi(t)} \leq 1, \qquad for\ each\ t \in [0,1],$$

and

$$\lim_{t \to 0} \frac{t}{\phi(\sqrt{t})} = 0$$

Proof. We know that the function $\phi(t)/t$ decreases and it is sufficient to prove the first assertion. The second one follows from the inequalities

$$\frac{t}{\phi(\sqrt{t})} = \frac{t}{\sqrt{t}} \frac{\sqrt{t}}{\phi(\sqrt{t})} \leq \sqrt{t}$$

■

Proposition 67 *Fix* $\psi, \phi \in \Phi[0,1]$*with* $\psi \geq \phi$. *If* $f \in Hol_{\phi}[0,1]$, *then* $f \in Hol_{\psi}[0,1]$ *and* $\theta_{\psi}(t) \leq \theta_{\phi}(f)$.

Proof. It follows from the definition of $.\theta_{\phi}(f,t)$ that

$$\theta_{\psi}(f,1) = \sup_{0<h\leq 1} \frac{\phi(h)}{\psi(h)} \frac{\varpi(f,h)}{\phi(h)} \leq \sup_{0<h\leq 1} \frac{\omega(f,h)}{\phi(h)} = \theta_{\phi}(f,1)$$

■

Proposition 68 *Fix* $\psi, \phi \in \Phi[0,1]$ *with* $\psi \geq \phi$. *If* $0 \leq r \leq m$, *then* $C_{\phi}^{m}[0,1] \subset C_{\psi}^{r}[0,1]$.

Theorem 69 *Assume* $0 \leq r < m$,$\phi \in \Phi[0,1]$,*and* $f \in C_{\phi}^{m}[0,1]$
(*i*) *If* $n \geq \max\{r+2, r(r+1)\}$, *then*

$$\|f - B_n(f)\|_{r,0} \leq \frac{r(r-1)}{2n}\|D^r f\| + \frac{r}{2n}\left\|D^{r+1}f\right\|$$
$$+\frac{1}{4\sqrt{n}}\varpi\left(f', \frac{1}{\sqrt{n}}\right) + \frac{\phi(1/\sqrt{n})}{2\sqrt{n}}\varpi_{\phi}\left(D^{r+1}f, \frac{1}{\sqrt{n}}\right)$$

(*ii*) *if* $\psi \in \Phi[0,1]$, $\phi \leq \psi$ *and* $n \geq \max\{r+2, r(r+1)\}$, *then*

$$\|f - B_n(f)\|_{r,\phi} \leq \frac{r(r-1)}{2n}\|D^r f\| + \frac{r}{2n}\left\|D^{r+1}f\right\|$$

$$+\frac{1}{4\sqrt{n}}\varpi\left(f', \frac{1}{\sqrt{n}}\right) + \frac{7\phi(1/\sqrt{n})}{4}\theta_{\phi}\left(D^{r+1}f, \frac{1}{\sqrt{n}}\right).$$

Chapter 4

Bernstein Polynomials and Numerical Integration

Proposition 70 *Fix* $a \in (0,1)$ *and* $n \in \mathbb{N}$.
if $n > 1$,*and* $na \notin \mathbb{N}$ *then*

$$B_n(sgn(t-a), x) = -1 + 2n \int_0^x p_{n-1,[na]} s ds = 1 - 2n \int_x^1 p_{n-1,[na]} s ds$$

and

$$\sum_{k=0}^{[na]} p_{n,k}(x) = n \int_x^1 p_{n-1,[na]} s ds$$

Proof. *Assume* $n > 1$. *Set* $j = [na]$ *and* $q_a(t) = sgn(t-a)$.
Since $j/n < a < (j+1)/n$,

$$\begin{aligned}
B_n'(sgn(t-a), x) &= n \sum_{k=0}^{n-1} \left(q_n\left(\frac{k+1}{n}\right) - q_a\left(\frac{k}{n}\right) \right) \left(\begin{array}{c} n-1 \\ k \end{array} \right) x^k (1-x)^{n-1-k} \\
&= n \left(\sum_{k=0}^{j-1} + \sum_{k=j}^{n-1} \right) \left(q_a\left(\frac{k+1}{n}\right) - q_a\left(\frac{k}{n}\right) \right) \left(\begin{array}{c} n-1 \\ k \end{array} \right) x^k (1-x)^{n-1-k} \\
&= 2n \left(\begin{array}{c} n-1 \\ j \end{array} \right) x^j (1-x)^{n-1-j} = 2n p_{n-1,j}(x) > 0.
\end{aligned}$$

If we take into account that $B_n(sgn(t-a), 0) = -1$,*then*

$$B_n(sgn(t-a),x) = -1 + 2n\int_0^x p_{n-1,[na]}sds = 1 - 2n\int_x^1 p_{n-1,[na]}sds.$$

On the other hand

$$\begin{aligned} B_n(sgn(t-a),x) &= -\sum_{k<na} p_{n,k}(x) + \sum_{k>na} p_{n,k}(x) \\ &= -1 + 2\sum_{k=1+[na]}^{n} p_{n,k}(x) = 1 - 2\sum_{k=0}^{[na]} p_{n,k}(x). \end{aligned}$$

■

By combining theses two representations one obtains the last assertion in proposition 72.

Proposition 71 *For each $n > 1$ and $0 \le k \le n$,*

$$\int_0^1 p_{n,k}(x)dx = \frac{1}{n+1}.$$

Proof. Fix k,$0 \le k < n$ and $a \in (0,1)$ such that $an \notin \mathbb{N}$ and $[an] = k$. From Proposition [70] one has

$$1 = sgn(1-a) = B_{n+1}(sgn(t-a),1) = -1 + 2(n+1)\int_0^1 p_{n,k}(s)ds.$$

For $k = n$

$$\int_0^1 p_{n,n}(s)ds = \int_0^1 x^n dx = \frac{1}{n+1}.$$

■

Theorem 72 *For each $f \in C[0,1]$ and every $n \in \mathbb{N}$ one has*

$$\|f - B_n f\| \le \omega_2\left(f, \frac{1}{\sqrt{n}}\right).$$

Theorem 73 *Assume* $f \in C^1[0,1]$ *and* $|f'(a) - f'(b)| \leq M|a-b|$ *for any* $a, b \in [0,1]$ *. For each* $n \in \mathbb{N}$ *and* $x \in [0,1]$*, then*

$$|B_n(f,x) - f(x)| \leq \frac{Mx(1-x)}{2n}$$

Proof. We know that f' is absolutely continuous and $\|f''\|_\infty \leq M$. For each $x; t \in [0,1]$,

$$\begin{aligned} |f(x) - f(t) - f'(x)(x-t)| &= \left|\int_t^x (t-s)f''(s)ds\right| \\ &\leq M\left|\int_t^x (t-s)ds\right| = \frac{M(x-t)^2}{2}. \end{aligned}$$

Therefore

$$|B_n(f,x) - f(x)| \leq \frac{M}{2}B_n((e_1-x)^2, x) = \frac{Mx(1-x)}{2n}$$

There is a more general result ■

4.1 Numerical integration

Proposition 74 *If* $f \in C[0,1]$ *and* $n \in \mathbb{N}$, *then*

$$\left|\int_0^1 f(x)dx - \frac{1}{n+1}\sum_{k=0}^n f\left(\frac{k}{n}\right)\right| \leq \omega_2\left(f, \frac{1}{\sqrt{n}}\right).$$

Proof. From proposition 73 and Theorem 74, we know that

$$\begin{aligned} \left|\int_0^1 f(x)dx - \frac{1}{n+1}\sum_{k=0}^n f\left(\frac{k}{n}\right)\right| &= \left|\int_0^1 f(x)dx - \sum_{k=0}^n \int_0^1 f\left(\frac{k}{n}\right)p_{n,k}(x)dx\right| \\ &= \left|\int_0^1 (f(x) - B_n(f,x))dx\right| \leq \omega_2\left(f, \frac{1}{\sqrt{n}}\right) \end{aligned}$$

■

There is a better estimate for twice differentiable functions.

Proposition 75 *If $f \in C^2[0;1]$ and $n \in \mathbb{N}$, then*

$$\left| \int_0^1 f(x)dx - \frac{1}{n+1} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \right| \leq \frac{1}{12n} \|f''\| .$$

Proof. We use Theorem 75 to obtain

$$\left| \int_0^1 (f(x) - B_n(f,x))dx \right| \leq \frac{\|f''\|}{2n} \int_0^1 x(1-x)dx = \frac{1}{12n} \|f''\| .$$

■

Example 76 *Let $f(x) = x^2$.and $n = 100$*

we have

$$I(f) = \int_0^1 f(x)dx = \int_0^1 x^2 dx = \frac{1}{3}.$$

$$I_a(f) = \frac{1}{n+1} \sum_{k=0}^{n} f\left(\frac{k}{n}\right)$$

$$\simeq \frac{1}{100} \sum_{k=0}^{100} \left(\frac{k}{100}\right)^2$$

$$\simeq 0.335$$

$$E = |I(f) - I_a(f)|$$

$$\left| \frac{1}{3} - 0.335 \right| = 0.0016666667$$

and we have

$$\begin{aligned} \frac{1}{12n} \|f''\| &= \frac{1}{12n} \|2\| = \frac{2}{1200} \\ &= 0.0016666667. \end{aligned}$$

so we get

$$\left| \int_0^1 f(x)dx - \frac{1}{n+1} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \right| = 0.0016666667 \simeq \frac{1}{12n} \|f''\|$$

4.2 Some Problems

There is many problems unsolved yet have relation with bernstein operators, In my phd theme I will inchallah study other problems like

Problem 77 *Kac presented a representation of Bernstein polynomials by means of some independent functions. He showed how to obtain first two, but we are not able to construct the other ones*

Problem 78 *We do not know a good estimate for the constant C in an inequality of the form*

$$|B_n(f, x) - f(x)| \leq C\omega\left(f, \sqrt{\frac{x(x-1)}{n}}\right)$$

Problem 79 *Does there exist a bounded function $f : [0,1] \to \mathbb{R}$ such that the family $\{B_n(f)\}$ forms a sequence of orthogonal polynomials?*

Problem 80 *Find the saturation class of Bernstein polynomials in L_1.*

conclusion

The human being ignoes more than what he knows, and if he know something it's not necessary that he knows it from all sides besides it does not creat any lack of balance in what he knows.
From this idea, I tried through this humble work to give an overview about the Bernstein operators and giving some of its diffrent properties.
we refer to the benifit of this memory is not limited only in what has been said above, its beyond that, because the work was an opportunityto train on many programs and the way of doing researchs.
In conclusion, I hope I was able to present this research and I wish this work will be a begining for more deep and specialized reserches and studies in this field.

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الخلاصة

تعرضنا في هذه المذكرة الى موضوع مؤثرات برنشتاين وقدمنا بعض خواصه كما استعملنا نتائج هذا المؤثر في حساب التكاملات العددية .
الكلمات المفتاحية: بناخ، هيلبرت، مؤثر برنشتاين، المؤثرات الموجبة

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Abstract

we discuss In this memory the subject of bernstein operators and introduce some of their properties, also the application of this operator in nemurical integration.

Key word : Banach, Hilbert, Bernstein operator, positive operator.

Résumé

Dans ce mimoire, on a étudie l’opérateur de bernstein et exhibit quelque properiétés, suivi par l’application de cet opérateur dans l’intégration numérique.

Mots clé :Banach, Hilbert, opérateur de Bernstein, opérateur positive.